LAU
School of Arts and Sciences

LEBANESE AMERICAN UNIVERSITY
Department of Computer Science and Mathematics

MTH102 – Calculus II
Spring 2015
Exam 1
(February 24, 2015)

NAME: Answer Key **ID:**

Duration: 70 minutes

Instructor: Ms. Liwa Sleiman

This exam is comprised of 8 problems.
Answer the questions in the space provided for each problem.
If more space is needed, use the back of the page.
Make sure to justify all your answers.
Graphic calculators are NOT allowed.

Problem	Grade points	
I	06 %	
II	12 %	
III	16 %	
IV	06 %	
V	08 %	
VI	12 %	
VII	24 %	
VIII	16 %	
Total	**100**	

I. a) (3 %) Express the following sum in sigma notation

$$1-\frac{1}{4}+\frac{1}{9}-\frac{1}{16}+\frac{1}{25} = \sum_{K=1}^{5}\left[\frac{1}{K^2}(-1)^{K+1}\right]$$

1 1 1

b) (3 %) Write the sum without sigma notation

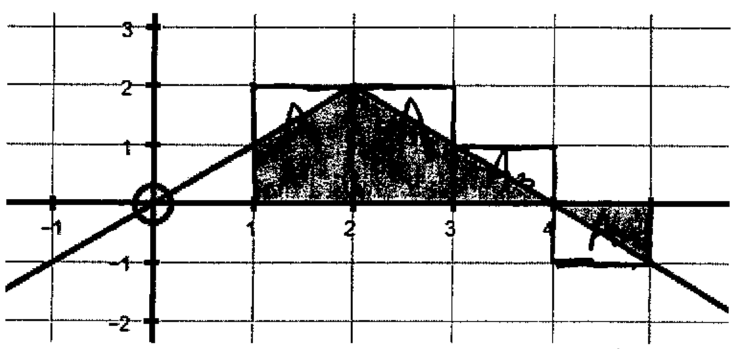
$$\sum_{k=3}^{4}\left(\sin\frac{k\pi}{2}-\cos k\pi\right)$$

$$= \left(\sin\frac{3\pi}{2}-\cos 3\pi\right)+\left(\sin\frac{4\pi}{2}-\cos 4\pi\right) \quad 1\tfrac{1}{2}$$

$$= -1+1+0-1 = -1 \quad 1\tfrac{1}{2}$$

II. (12 %)
Estimate the shaded area using **upper sum** with **four rectangles of equal width**

$f(x) = 2-|x-2|$ over the interval $[1, 5]$



$$\text{width} = \frac{b-a}{n} = \frac{5-1}{4} = \frac{4}{4} = 1 \quad 2$$

$$A_1 = f(2)\cdot 1 = 2 \quad 2$$

$$A_2 = f(2)\cdot 1 = 2 \quad 2$$

$$A_3 = f(3)\cdot 1 = 1 \quad 2$$

$$A_4 = |f(5)|\cdot 1 = |-1| = 1 \quad 2$$

$$\text{Estimated Area} = A_1 + A_2 + A_3 + A_4 \quad 2$$

$$= 2+2+1+1$$

$$= 6 \text{ units}^2$$

III. Consider the function $f(x) = 4x^3 + 3x^2 + 2x + 2$ over the interval [0, 1]

a) (12 %)
Find a formula for the **Riemann sum** obtained by dividing the interval into **n equal subintervals** and using the **right-hand endpoint** for each c_k. Then take a **limit of these sums as n → ∞** to calculate the area under the curve over [0, 1].

Hint:

$$\sum_{k=1}^{n} 1 = n \; ; \; \sum_{k=1}^{n} k = \frac{n(n+1)}{2} \; ; \; \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} \; ; \; \sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2$$

b) (4 %)
Find the area under the curve over [0, 1] using definite integral.

a) $\sum_{K=1}^{n} A_K$ 2 $= \sum_{K=1}^{n} f\left(a + k\frac{b-a}{n}\right) \cdot \frac{b-a}{n}$ $a=0$, $b=1$

1 $= \sum_{K=1}^{n} f\left(\frac{k}{n}\right) \cdot \frac{1}{n}$

1 $= \sum_{K=1}^{n} \left[4\left(\frac{k}{n}\right)^3 + 3\left(\frac{k}{n}\right)^2 + 2\left(\frac{k}{n}\right) + 2\right] \cdot \left[\frac{1}{n}\right]$

1 $= \sum_{K=1}^{n} \left(\frac{4K^3}{n^3} + \frac{3K^2}{n^2} + \frac{2k}{n} + 2\right) \cdot \frac{1}{n}$

1 $= \sum_{K=1}^{n} \left(\frac{4K^3}{n^4} + \frac{3K^2}{n^3} + \frac{2K}{n^2} + \frac{2}{n}\right)$

1 $= \frac{4}{n^4}\left(\sum_{K=1}^{n} K^3\right) + \frac{3}{n^3}\left(\sum_{K=1}^{n} K^2\right) + \frac{2}{n^2}\left(\sum_{K=1}^{n} K\right) + \frac{2}{n}\left(\sum_{K=1}^{n} 1\right)$

1 $= \frac{4}{n^4}\left[\frac{n(n+1)}{2}\right]^2 + \frac{3}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right] + \frac{2}{n^2}\left[\frac{n(n+1)}{2}\right] + \frac{2}{n}(n)$

1 $= \frac{4n^2(n+1)^2}{4n^4} + \frac{3n(n+1)(2n+1)}{6n^3} + \frac{2n(n+1)}{2n^2} + 2$

$\text{Area} = \lim_{n\to\infty} \sum_{K=1}^{n} A_K$ 1 $= \lim_{n\to\infty}$ 1 $\frac{4n^4}{4n^4} + \frac{6n^3}{6n^3} + \frac{2n^2}{2n^2} + 2 = 1+1+1+2$ 1
$= 5 \text{ units}^2$

b) $\text{Area} = \int_0^1 (4x^3 + 3x^2 + 2x + 2)\,dx$ 2 $= \left[\frac{4x^4}{4} + \frac{3x^3}{3} + \frac{2x^2}{2} + 2x\right]_0^1$ 1
$= (1+1+1+2) - 0 = 5 \text{ units}^2$ 1

IV. (6 %)

Find the average value of $f(x) = \sin 4x$ over the interval $\left[0, \frac{\pi}{4}\right]$.

$$\text{Avg}(f) = \frac{1}{b-a}\int_a^b f(x)\,dx \quad 1$$

$$= \frac{1}{\frac{\pi}{4}-0}\int_0^{\frac{\pi}{4}} \sin 4x\,dx \quad 1$$

$$= \frac{4}{\pi}\left[\frac{-\cos 4x}{\cancel{4}}\right]_0^{\pi/4} \quad 2$$

$$= \frac{-1}{\pi}\left(\cos\frac{4\pi}{4} - \cos 0\right) = \frac{-1}{\pi}(-1-1) = \frac{2}{\pi}$$

$1 \quad 1$

V. (8 %)

Find the total area between the curves $y = x^3 - x + 1$ and $y = 1$.

$$y = y \quad \tfrac{1}{2}$$

$$x^3 - x + 1 = 1$$

$$x^3 - x = 0 \quad \tfrac{1}{2}$$

$$x(x^2-1) = 0$$

$$x(x-1)(x+1) = 0 \quad \tfrac{1}{2}$$

$$x = 0 \quad x = 1 \quad x = -1 \quad \tfrac{1}{2}$$

$$\tfrac{1}{2}\quad \int_{-1}^{0}(x^3 - x + 1 - 1)\,dx = \int_{-1}^{0}(x^3 - x)\,dx = \left[\frac{x^4}{4} - \frac{x^2}{2}\right]_{-1}^{0} \quad \tfrac{1}{2}$$

$\tfrac{1}{2}$

$$= 0 - \left(\frac{1}{4} - \frac{1}{2}\right) = \frac{1}{4} \quad \tfrac{1}{2}$$

$$\tfrac{1}{2}\quad \int_0^1 (x^3 - x + 1 - 1)\,dx = \int_0^1 (x^3 - x)\,dx = \left[\frac{x^4}{4} - \frac{x^2}{2}\right]_0^1 \quad \tfrac{1}{2}$$

$\tfrac{1}{2}$

$$= \left(\frac{1}{4} - \frac{1}{2}\right) - 0 = -\frac{1}{4} \quad \tfrac{1}{2}$$

$$\text{Area} = \left|\frac{1}{4}\right| + \left|\frac{-1}{4}\right| = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}\ \text{units}^2$$

$1 \quad \tfrac{1}{2} \quad \tfrac{1}{2}$

VI. Use the Fundamental Theorem of Calculus to find $\frac{dy}{dx}$

a) (6 %) $y = \int_x^{\sin x} \sqrt{5t-1}\,dt$

$$y = \int_x^0 \sqrt{5t-1}\,dt + \int_0^{\sin x} \sqrt{5t-1}\,dt \quad 1\tfrac{1}{2}$$

$$= -\int_0^x \sqrt{5t-1}\,dt + \int_0^{\sin x} \sqrt{5t-1}\,dt \quad 1\tfrac{1}{2}$$

$$\frac{dy}{dx} = \underbrace{-\sqrt{5x-1}}_{1\frac{1}{2}} + \left(\underbrace{\sqrt{5\sin x - 1}}_{1\frac{1}{2}} \cdot \cos x\right)$$

b) (6 %) $y = x\int_\pi^{\sqrt{x}} \cos t\;dt$

$$y = uv$$

$$\frac{dy}{dx} = (uv)' = u'v + v'u \quad \tfrac{1}{2}$$

$\tfrac{1}{2}$ $u = x$, $u' = 1$

$$v = \int_\pi^{\sqrt{x}} \cos t\,dt \;\tfrac{1}{2} = \left[\sin t\right]_\pi^{\sqrt{x}} \;\tfrac{1}{2} = \sin\sqrt{x} \;\tfrac{1}{2} - \sin\pi = \sin\sqrt{x} \;\tfrac{1}{2}$$

$$v' = \frac{d}{dx}\int_\pi^{\sqrt{x}} \underbrace{\cos t}_{1}\,dt = \underbrace{\cos\sqrt{x}}_{1} \cdot \frac{1}{2\sqrt{x}}$$

$$\frac{dy}{dx} \overset{1}{=} \sin\sqrt{x} + \cos\sqrt{x} \cdot \frac{1}{2\sqrt{x}} \cdot x$$

$\boxed{\text{OR}}$

$$\frac{dy}{dx} = \int_\pi^{\sqrt{x}} \cos t\,dt + \cos\sqrt{x} \cdot \frac{1}{2\sqrt{x}} \cdot x \quad 4$$

$$= \left[\sin t\right]_\pi^{\sqrt{x}} + \cos\sqrt{x} \cdot \frac{1}{2\sqrt{x}} \cdot x \quad 1$$

$$= \sin\sqrt{x} - \sin\pi + \cos\sqrt{x} \cdot \frac{1}{2\sqrt{x}} \cdot x \quad \tfrac{1}{2}$$

$$= \sin\sqrt{x} + \cos\sqrt{x} \cdot \frac{1}{2\sqrt{x}} \cdot x \quad \tfrac{1}{2}$$

VII. Evaluate the following integrals

a) (8 %) $\int_0^2 \frac{4}{\sqrt{4x+1}}\, dx = \int_0^2 4\,(4x+1)^{-1/2}\, dx$ 1

let $u = 4x+1$ 1

$du = 4\,dx$ 1

$= \int_{x=0}^{x=2} u^{-1/2}\, du$ 1 $= \left[\frac{u^{1/2}}{1/2}\right]_{x=0}^{x=2}$ 1 $= 2\left[(4x+1)^{1/2}\right]_0^2$ 1

1 $= 2\,(9^{1/2} - 1) =$ 1 $2(3-1) = 4$

b) (8 %) $\int \frac{1}{x^2}\left(\frac{1-x}{x}\right)^4 dx$

$= -\int -\frac{1}{x^2}\left(\frac{1}{x} - 1\right)^4 dx$ 2

let $u = \frac{1}{x} - 1$ 1

$du = \frac{-1}{x^2}\, dx$ 1½

$= -\int u^4\, du$ 1½

$= -\frac{u^5}{5} + c$ 1

$= -\frac{1}{5}\left(\frac{1}{x} - 1\right)^5 + c$ 1

c) (8 %) $\frac{1}{5}\int 5\tan^3(5x)\,.\sec^2(5x)\ dx$

let $u = \tan(5x)$ 2

$du = 5$ 1 $\sec^2(5x)$ 1 dx

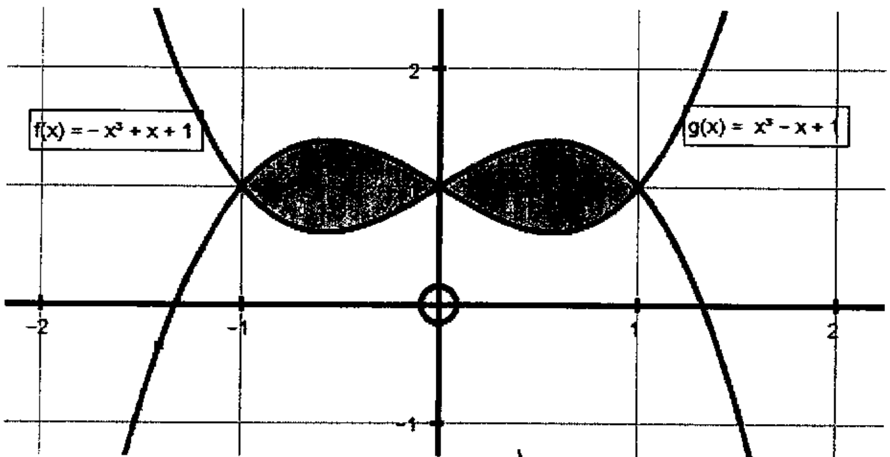
$= \frac{1}{5}\int u^3\, du$ 2

$= \frac{1}{5}\,\frac{u^4}{4} + c$ 1

$= \frac{1}{20}\tan^4(5x) + c$ 1

VIII. Write the total area of the shaded region as definite integral
(DO NOT evaluate)

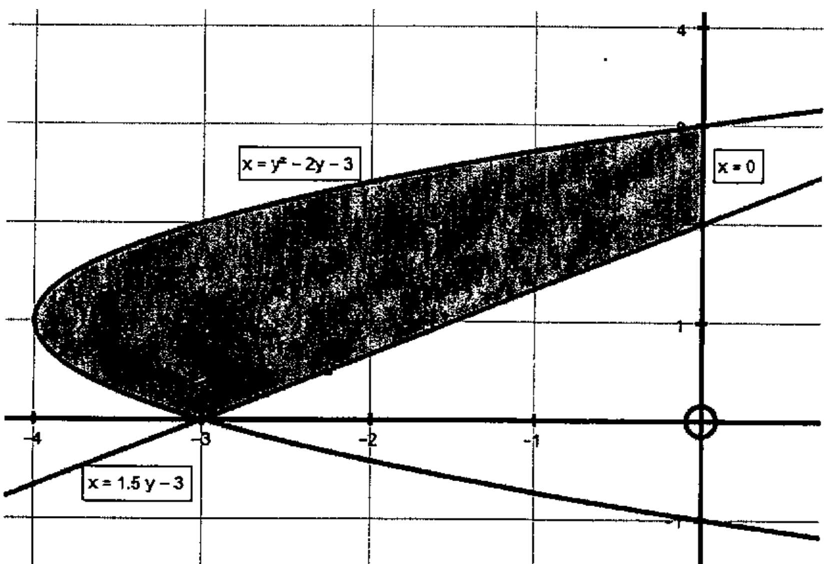
a) (8 %) the region enclosed by
$f(x) = -x^3 + x + 1$ and $g(x) = x^3 - x + 1$



$$A = \int_{-1}^{0} \underbrace{\left(g(x) - f(x)\right)}_{\text{(upper - lower)}} dx + \int_{0}^{1} \underbrace{\left(f(x) - g(x)\right)}_{\text{(upper - lower)}} dx$$

$1\frac{1}{2}$ $1\frac{1}{2}$ 1 $1\frac{1}{2}$ $1\frac{1}{2}$ 1

b) (8 %) the region enclosed by
$x = y^2 - 2y - 3$ and $x = 1.5\,y - 3$ and $x = 0$



$$A = \int_{0}^{2} \underbrace{(1.5y - 3)}_{\text{right}} - \underbrace{(y^2 - 2y - 3)}_{\text{left}}\, dy + \int_{2}^{3} \underbrace{0}_{\text{right}} - \underbrace{(y^2 - 2y - 3)}_{\text{left}}\, dy$$

$1\frac{1}{2}$ $1\frac{1}{2}$ 1 $1\frac{1}{2}$ $1\frac{1}{2}$ 1